



Change of basis

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Table of contents

01

Invertible Linear
Maps

02

Basis Review

03

Change of Basis

04

$L(V)$ and Change
of Basis

05

$L(V,W)$ and Change of
Basis



02

Basis Review



Review: Basis

Example

Find the coordinate vector of $2 + 7x + x^2 \in \mathcal{P}^2$ with respect to the basis $B = \{x + x^2, 1 + x^2, 1 + x\}$.

If $C = \{1, x, x^2\}$ is the standard basis of \mathcal{P}^2 then we have $[2 + 7x + x^2]_C = (2, 7, 1)$.

Solution

We want to find scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$2 + 7x + x^2 = c_1(x + x^2) + c_2(1 + x^2) + c_3(1 + x).$$

By matching coefficients of powers of x on the left-hand and right-hand sides above, we arrive at following system of linear equations:

$$c_2 + c_3 = 2$$

$$c_1 + c_3 = 7$$

$$c_1 + c_2 = 1$$

This linear system has $c_1 = 3, c_2 = -2, c_3 = 4$ as its unique solution, so our desired coordinate vector is

$$[2 + 7x + x^2] = (c_1, c_2, c_3) = (3, -2, 4)$$

03

Change of Basis



Introduction to change of basis

- $B = \{v_1, \dots, v_n\}$ are basis of \mathbb{R}^n .



- $P = [v_1 \ v_2 \ \dots \ v_n]$

- $P[a]_B = a$



Change of Basis

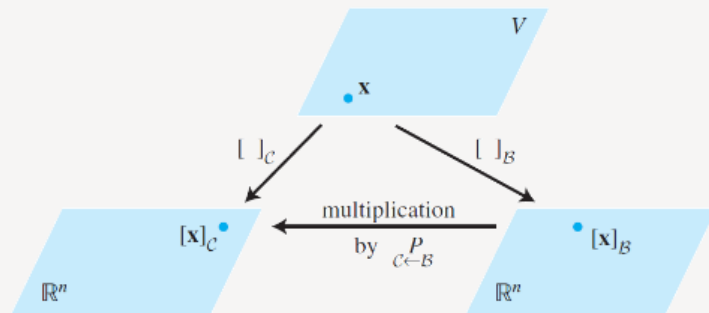
Theorem

Let $B = \{b_1, b_2, \dots, b_n\}$ and $C = \{c_1, c_2, \dots, c_n\}$ be bases of a vector space V . Then there is a unique $n \times n$ matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B} [x]_B$$

The columns of $P_{C \leftarrow B}$ are the C -coordinate vectors of the vectors in basis B . That is,

$$P_{C \leftarrow B} = [[b_1]_C \quad [b_2]_C \quad \dots \quad [b_n]_C]$$



$$(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$$

$$P_B[x]_B = x, \quad P_C[x]_C = x, \quad \text{and} \quad [x]_C = P_C^{-1}x$$

$$[x]_C = P_C^{-1}x = P_C^{-1}P_B[x]_B$$

Change of basis

Example

Find the change-of-basis matrices $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$ for the bases
 $B = \{x + x^2, 1 + x^2, 1 + x\}$ and $C = \{1, x, x^2\}$
of \mathcal{P}^2 .

Then find the coordinate vector of $2 + 7x + x^2$ with respect to B.

Change of basis

Example

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, the bases for \mathbb{R}^2 given by $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$.

Find the change-of-coordinates matrix from C to B.

Find the change-of-coordinates matrix from B to C.

Change of basis

Example

Find the change-of-basis matrix $P_{C \leftarrow B}$, where

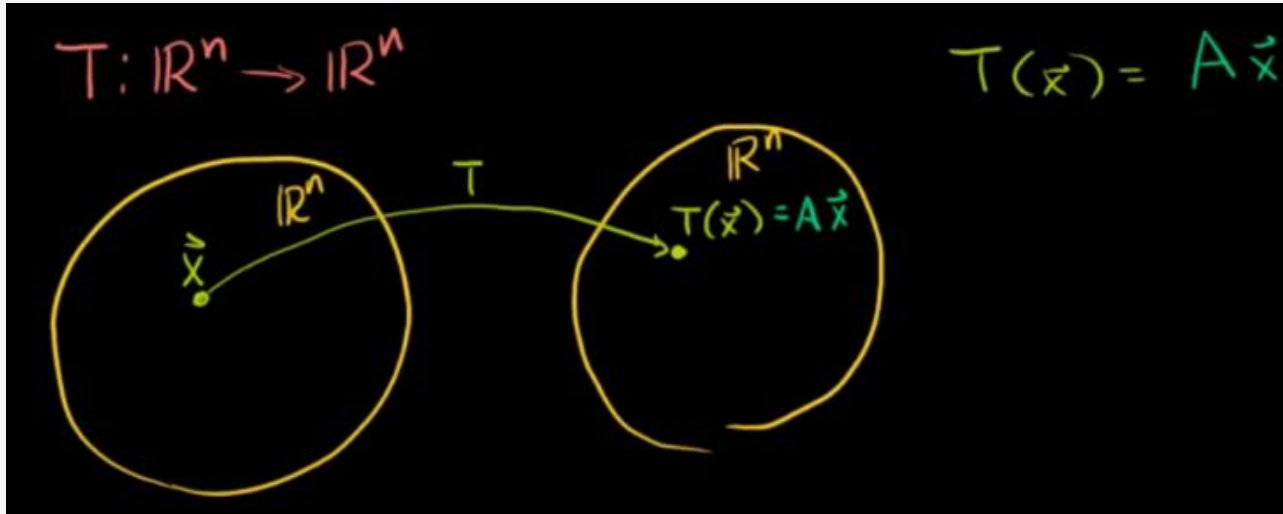
$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

04

$L(V)$ and Change of Basis

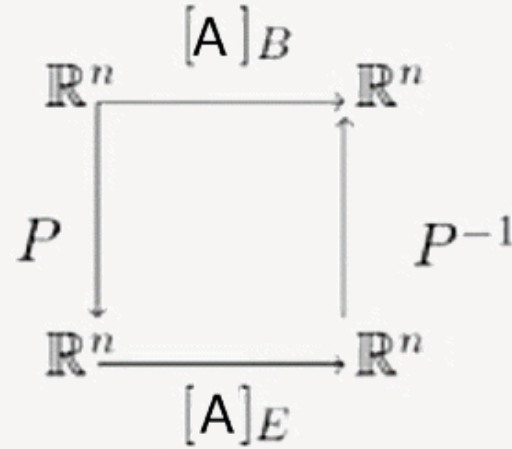


Transformation with change of basis



- $B = \{v_1, v_2, \dots, v_n\}$ are basis of \mathbb{R}^n .
- $P = [v_1 \ v_2 \ \dots \ v_n]$
- $[T(x)]_B = P^{-1}AP[x]_B$

Change of basis



$$[A]_B = P^{-1}[A]_E P$$

05

$L(V, W)$ & Change of Basis



Matrix representation of linear function

A linear transformation which looks complex with respect to one basis can become much easier to understand when you choose the correct basis.

Important

Let $T: V \rightarrow W$ be a linear function and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in V$ where $E = \{e_1, \dots, e_n\}$, $B = \{b_1, \dots, b_m\}$ are basis of V, W .

$$u = c_1 e_1 + \dots + c_n e_n \rightarrow T(u) = c_1 T(e_1) + \dots + c_n T(e_n)$$

$$T(u) = d_1 b_1 + \dots + d_m b_m$$

$$[T(u)]_B = [[T(e_1)]_B, \dots, [T(e_n)]_B][T(u)]_E$$


Linear Transformation

Example

We have $B = \{x^3, x^2, x, 1\}$ and $B' = \{x^2, x, 1\}$ are bases for $\mathcal{P}_3(x)$ and $\mathcal{P}_2(x)$, respectively. Find the matrix of transformation $T: \mathcal{P}_3(x) \rightarrow \mathcal{P}_2(x)$.

Solution

Since $\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$ the vector representation of $a_3x^3 + a_2x^2 + a_1x + a_0 \in \mathcal{P}^3(x)$, we have


$$\begin{aligned} \left[\frac{d}{dx} \right]_{\{B, B'\}} &= \begin{bmatrix} \frac{d}{dx}(x^3) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x) & \frac{d}{dx}(1) \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$



Isomorphisms

Example

Show that the vector space $V = \text{span}(e^x, xe^x, x^2e^x)$ and \mathbb{R}^3 are isomorphic.

The standard way to show that two space are isomorphic is to construct an isomorphism between them. To this end, consider the linear transformation $T: \mathbb{R}^3 \rightarrow V$ defined by

$$T(a, b, c) = ae^x + bxe^x + cx^2e^x.$$

It is straightforward to show that this function is linear transformation, so we just need to convince ourselves that it is invertible. We can construct the standard matrix $[T]_{B \leftarrow E}$, where $E = \{e_1, e_2, e_3\}$ is the standard basis of \mathbb{R}^3 :

$$\begin{aligned} [T]_{B \leftarrow E} &= [[T(1, 0, 0)]_B, [T(0, 1, 0)]_B, [T(0, 0, 1)]_B] \\ &= [[e^x]_B, [xe^x]_B, [x^2e^x]_B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Since $[T]_{B \leftarrow E}$ is clearly invertible (the identity matrix is its own inverse), T is invertible too and is thus an isomorphism.